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ON THE POSITIVE PELL EQUATION $y^2 = 12x^2 + 4$

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ABSTRACT

The binary quadratic Diophantine equation represented by $y^2 = 12x^2 + 4$ is analyzed for its non-zero distinct integer solutions. Employing the lemma of Brahmagupta, infinitely many integral solutions of the above Pell equation are obtained. The recurrence relations on the solutions are also presented. A few interesting relations between the solutions and special number patterns namely, Polygonal numbers are also given. Further, employing the solutions of the above equation, we have obtained solutions of other choices of hyperbolas and parabolas.

Keywords: *Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.*
Mathematics subject Classification(2010):11D09

I. INTRODUCTION

It is well known that the Pell equation $x^2 - Dy^2 = \pm 1 (D > 0$ and square free) has always positive integer solutions.[1-3] When $N \neq 1$, the pell equation $x^2 - Dy^2 = N$ may not have any positive integer solutions. For example the equations $x^2 = 3y^2 - 1$ and $x^2 = 7y^2 - 4$ have no positive integer solutions. When K is a positive integer and $D \in \{K^2 \pm 4, K^2 \pm 1\}$, positive integer solutions of the equation $x^2 - Dy^2 = \pm 4$ and $x^2 - Dy^2 = \pm 1$have been investigated by Jones in [4]. For an extensive review of various problems, one may refer[5-15]. In this communication, yet another interesting equation given by $y^2 = 12x^2 + 4$ is considered and infinitely many integer solutions are obtained. The recurrence relations on the solutions are also given. A few interesting relations between the solutions and special numbers are presented.

II. NOTATIONS

$t_{m,n} = n\left(1 + \frac{(n-1)(m-2)}{2}\right)$ polygonal number of rank n with size m

$p^m = \frac{1}{6} n(n+1)((m-2)n + 5 - m)$ Pyramidal number of rank n with size m

III. METHODS OF ANALYSIS

The Diophantine equation to be solved for its non-zero distinct integral solution is,

$$y^2 = 12x^2 + 4 \qquad (1)$$

The smallest positive integer solutions of (1) is,

$x_0 = 1$, $y_0 = 4$ $D = 12$

Now, consider the Pell equation is

$$y^2 = 12x^2 + 1 \qquad (2)$$

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Whose fundamental solution is

$\tilde{x}_0 = 2$, $\tilde{y}_0 = 7$,

The other solution of (2) can be derived from the relations,

$$\tilde{x}_n = \frac{1}{2\sqrt{12}} g_n, \quad \tilde{y}_n = \frac{1}{2} f_n$$

Where,

$$f_n = (7+2\sqrt{12})^{n+1} + (7-2\sqrt{12})^{n+1}$$
$$g_n = (7+2\sqrt{12})^{n+1} - (7-2\sqrt{12})^{n+1} \quad n = 0,1,2......$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by,

$$x_{n+1} = \frac{1}{2} f_n + \frac{2}{\sqrt{12}} g_n$$
$$y_{n+1} = 2 f_n + \frac{6}{\sqrt{12}} g_n$$

The recurrence relations satisfied by the solution x and y are given by,

$$x_{n+3} - 14x_{n+2} + x_{n+1} = 0$$
$$y_{n+3} - 14y_{n+2} + y_{n+1} = 0, \quad n = 0,1,2......$$

Some numerical examples of x_n and y_n satisfying (1) are given in the Table 1 below,

Table 1: Examples

n	x_n	y_n
0	1	4
1	15	52
2	209	724
3	2911	10084
4	40545	140452

From the above table, we observe some interesting relations among the solutions which are presented below.

x_n values are odd and y_n values are even.

Each of the following expression is a nasty number:

$$12[1 + y_{2n+2} - 3x_{2n+2}]$$
$$6[2 + x_{2n+3} - 13x_{2n+2}]$$

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$$\frac{3}{7}[28+x_{2n+4}-181x_{2n+2}]$$

$$\frac{12}{7}[7+y_{2n+3}-45x_{2n+2}]$$

$$\frac{12}{97}[97+y_{2n+4}-627x_{2n+2}]$$

$$\frac{12}{7}[7+13y_{2n+2}-3x_{2n+3}]$$

$$\frac{12}{97}[97+181y_{2n+2}-3x_{2n+4}]$$

$$\frac{3}{2}[8+15y_{2n+2}-y_{2n+3}]$$

$$\frac{3}{28}[112+209y_{2n+2}-y_{2n+4}]$$

$$12+78x_{2n+4}-1086x_{2n+3}$$

$$12+156y_{2n+3}-540x_{2n+3}$$

$$\frac{12}{7}[7+13y_{2n+4}-627x_{2n+3}]$$

$$\frac{12}{7}[7+181y_{2n+3}-45x_{2n+4}]$$

$$12+2172y_{2n+4}-7524x_{2n+4}$$

$$\frac{3}{2}[8+209y_{2n+3}-15y_{2n+4}]$$

Each of the following expressions is a cubical integer.

$$2y_{3n+3}-6x_{3n+3}+6y_{n+1}-18x_{n+1}$$

$$x_{3n+4}-13x_{3n+3}+3x_{n+2}-39x_{n+1}$$

$$\frac{1}{14}[x_{3n+5}-181x_{3n+3}+3x_{n+3}-543x_{n+1}]$$

$$\frac{1}{7}[2y_{3n+4}-90x_{3n+3}+6y_{n+2}-270x_{n+1}]$$

$$\frac{1}{97}[2y_{3n+5}-1254x_{3n+3}+6y_{n+3}-3762x_{n+1}]$$

$$\frac{1}{7}[26y_{3n+3}-6x_{3n+4}+78y_{n+1}-18x_{n+2}]$$

$$\frac{1}{97}[362y_{3n+3}-6x_{3n+5}+1086y_{n+1}-18x_{n+3}]$$

$$\frac{1}{4}[15y_{3n+3}-y_{3n+4}+45y_{n+1}-3y_{n+2}]$$

$$\frac{1}{56}[209y_{3n+3}-y_{3n+5}+627y_{n+1}-3y_{n+3}]$$
$$13x_{3n+5}-181x_{3n+4}+39x_{n+3}-543x_{n+2}$$
$$26y_{3n+4}-90x_{3n+4}+78y_{n+2}-270x_{n+2}$$
$$\frac{1}{7}[26y_{3n+5}-1254x_{3n+4}+78y_{n+3}-3762x_{n+2}]$$
$$\frac{1}{7}[362y_{3n+4\backslash}-90x_{3n+5}+1086y_{n+2}-270x_{n+3}]$$
$$[362y_{3n+5\backslash}-1254x_{3n+5}+1086y_{n+3}-3762x_{n+3}]$$
$$\frac{1}{4}[209y_{3n+4\backslash}-15y_{3n+5}+627y_{n+2}-45y_{n+3}]$$

Each of the following expressions is a biquadratic integer.

$$2y_{4n+4}-6x_{4n+4}+8y_{2n+2}-24x_{2n+2}+6$$
$$x_{4n+5}-13x_{4n+4}+4x_{2n+3}-52x_{2n+2}+6$$
$$\frac{1}{14}[x_{4n+6}-181x_{4n+4}+4x_{2n+4}-724x_{2n+2}+84]$$
$$\frac{1}{7}[2y_{4n+5}-90x_{4n+4}+8y_{2n+3}-360x_{2n+2}+42]$$
$$\frac{1}{97}[2y_{4n+6}-1254x_{4n+4}+8y_{2n+4}-5016x_{2n+2}+582]$$
$$\frac{1}{7}[26y_{4n+4}-6x_{4n+5}+104y_{2n+2}-24x_{2n+3}+42]$$
$$\frac{1}{97}[362y_{4n+4}-6x_{4n+6}+1448y_{2n+2}-24x_{2n+4}+582]$$
$$\frac{1}{4}[15y_{4n+4}-y_{4n+5}+60y_{2n+2}-4y_{2n+3}+24]$$
$$\frac{1}{56}[209y_{4n+4}-y_{4n+6}+836y_{2n+2}-4y_{2n+4}+336]$$
$$13x_{4n+6}-181x_{4n+5}+52x_{2n+4}-724x_{2n+3}+6$$
$$26y_{4n+5}-90x_{4n+5}+104y_{2n+3}-360x_{2n+3}+6$$
$$\frac{1}{7}[26y_{4n+5}-1254x_{4n+6}+104y_{2n+3}-5016x_{2n+4}+42]$$
$$\frac{1}{7}[362y_{4n+5}-90x_{4n+6}+1448y_{2n+3}-360x_{2n+4}+42]$$
$$362y_{4n+6}-1254x_{4n+6}+1448y_{2n+4}-5016x_{2n+4}+6$$
$$\frac{1}{4}[209y_{4n+5}-15y_{4n+6}+836y_{2n+3}-60y_{2n+4}+24]$$

Each of the following expression is a quintic integer.

$$2y_{5n+5}-6x_{5n+5}+10y_{3n+3}-30x_{3n+3}+20y_{n+1}-60x_{n+1}$$

$$x_{5n+6}-13x_{5n+5}-5x_{3n+4}-65x_{3n+3}+10x_{n+2}-130x_{n+1}$$

$$\frac{1}{14}\left[x_{5n+7}-181x_{5n+5}+5x_{3n+5}-905x_{3n+3}+10x_{n+3}-1810x_{n+1}\right]$$

$$\frac{1}{7}\left[2y_{5n+6}-90x_{5n+5}+10y_{3n+4}-450x_{3n+3}+20y_{n+2}-900x_{n+1}\right]$$

$$\frac{1}{97}\left[2y_{5n+7}-1254x_{5n+5}+10y_{3n+5}-6270x_{3n+3}+20y_{n+3}-12540x_{n+1}\right]$$

$$\frac{1}{7}\left[26y_{5n+5}-6x_{5n+6}+130y_{3n+3}-30x_{3n+4}+260y_{n+1}-60x_{n+2}\right]$$

$$\frac{1}{97}\left[362y_{5n+5}-6x_{5n+7}+1810y_{3n+3}-30x_{3n+5}+3620y_{n+1}-60x_{n+3}\right]$$

$$\frac{1}{4}\left[15y_{5n+5}-y_{5n+6}+75y_{3n+3}-5y_{3n+4}+150y_{n+1}-10y_{n+2}\right]$$

$$\frac{1}{56}\left[209y_{5n+5}-y_{5n+7}+1045y_{3n+3}-5y_{3n+5}+2090y_{n+1}-10y_{n+3}\right]$$

$$13x_{5n+7}-181x_{5n+6}+65x_{3n+5}-905x_{3n+4}+130x_{n+3}-1810x_{n+2}$$

$$26y_{5n+6}-90x_{5n+6}+130y_{3n+4}-450x_{3n+4}+260y_{n+2}-900x_{n+2}$$

$$\frac{1}{7}\left[26y_{5n+7}-1254x_{5n+6}+130y_{3n+5}-6270x_{3n+4}+260y_{n+3}-12540x_{n+2}\right]$$

$$\frac{1}{7}\left[362y_{5n+6}-90x_{5n+7}+1810y_{3n+4}-450x_{3n+5}+3620y_{n+2}-900x_{n+3}\right]$$

$$362y_{5n+7}-1254x_{5n+7}+1810y_{3n+5}-6270x_{3n+5}+3620y_{n+3}-12540x_{n+3}$$

$$\frac{1}{4}\left[209y_{5n+6}-15y_{5n+7}+1045y_{3n+4}-75y_{3n+5}+2090y_{n+2}-150y_{n+3}\right]$$

Relations among the solutions are given below.

$$x_{n+2}=2y_{n+1}+7x_{n+1}$$

$$x_{n+3}=28y_{n+1}+97x_{n+1}$$

$$y_{n+2}=7y_{n+1}+24x_{n+1}$$

$$y_{n+3}=97y_{n+1}+336x_{n+1}$$

$$x_{n+3}=14x_{n+2}-x_{n+1}$$

$$2y_{n+2}=7x_{n+2}-x_{n+1}$$

$$2y_{n+3}=97x_{n+2}-7x_{n+1}$$

$$4y_{n+2}=x_{n+3}-x_{n+1}$$

$$28y_{n+3}=97x_{n+3}-x_{n+1}$$

$$7y_{n+3} = 97y_{n+2} + 24x_{n+1}$$
$$7x_{n+3} = 2y_{n+1} + 97x_{n+2}$$
$$7y_{n+2} = y_{n+1} + 24x_{n+2}$$
$$y_{n+3} = y_{n+1} + 48x_{n+2}$$
$$97y_{n+2} = 7y_{n+1} + 24x_{n+3}$$
$$97y_{n+3} = y_{n+1} + 336x_{n+3}$$
$$y_{n+3} = 14y_{n+2} - y_{n+1}$$
$$2y_{n+2} = x_{n+3} - 7x_{n+2}$$
$$2y_{n+3} = 7x_{n+3} - x_{n+2}$$
$$y_{n+3} = 7y_{n+2} + 24x_{n+2}$$
$$7y_{n+3} = y_{n+2} + 24x_{n+3}$$

IV. REMARKABLE OBSERVATIONS

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below:

Table 2: Hyperbola

S.NO	Hyperbola	(X,Y)
1	$Y^2 - 3X^2 = 4$	$(4x_{n+1} - y_{n+1}, 2y_{n+1} - 6x_{n+1})$
2	$4Y^2 - 3X^2 = 16$	$(15x_{n+1} - x_{n+2}, x_{n+2} - 13x_{n+1})$
3	$4Y^2 - 3X^2 = 3136$	$(209x_{n+1} - x_{n+3}, x_{n+3} - 181x_{n+1})$
4	$Y^2 - 3X^2 = 196$	$(52x_{n+1} - y_{n+2}, 2y_{n+2} - 90x_{n+1})$
5	$Y^2 - 4X^2 = 37636$	$(724x_{n+1} - y_{n+3}, 2y_{n+3} - 1254x_{n+1})$
6	$Y^2 - 3X^2 = 196$	$(4x_{n+2} - 15y_{n+1}, 26y_{n+1} - 6x_{n+2})$
7	$Y^2 - 3X^2 = 37636$	$(4x_{n+3} - 209y_{n+1}, 362y_{n+1} - 6x_{n+3})$
8	$3Y^2 - 4X^2 = 192$	$(y_{n+2} - 13y_{n+1}, 15y_{n+1} - y_{n+2})$
9	$3Y^2 - 4X^2 = 37632$	$(y_{n+3} - 181y_{n+1}, 209y_{n+1} - y_{n+3})$
10	$4Y^2 - 3X^2 = 16$	$(209x_{n+2} - 15x_{n+3}, 13x_{n+3} - 181x_{n+2})$
11	$Y^2 - 3X^2 = 4$	$(52x_{n+2} - 15y_{n+2}, 26y_{n+2} - 90x_{n+2})$
12	$Y^2 - 3X^2 = 196$	$(724x_{n+2} - 15y_{n+3}, 26y_{n+3} - 1254x_{n+2})$
13	$Y^2 - 3X^2 = 196$	$(52x_{n+3} - 209y_{n+2}, 362y_{n+2} - 90x_{n+3})$
14	$Y^2 - 3X^2 = 4$	$(724x_{n+3} - 209y_{n+3}, 362y_{n+3} - 1254x_{n+3})$
15	$3Y^2 - 4X^2 = 192$	$(13y_{n+3} - 181y_{n+2}, 209y_{n+2} - 15y_{n+3})$

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Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below:

Table 3: Parabola

S.NO	Parabola	(X,Y)
1	$Y-3X^2=4$	$(4x_{n+1}-y_{n+1}, 2y_{2n+2}-6x_{2n+2}+2)$
2	$4Y-3X^2=16$	$(15x_{n+1}-x_{n+2}, x_{2n+3}-13x_{2n+2}+2)$
3	$56Y-3X^2=3136$	$(209x_{n+1}-x_{n+3}, x_{2n+4}-181x_{2n+2}+28)$
4	$7Y-3X^2=196$	$(52x_{n+1}-y_{n+2}, 2y_{2n+3}-90x_{2n+2}+14)$
5	$97Y-3X^2=37636$	$(724x_{n+1}-y_{n+3}, 2y_{2n+4}-1254x_{2n+2}+194)$
6	$7Y-3X^2=196$	$(4x_{n+2}-15y_{n+1}, 26y_{2n+2}-6x_{2n+3}+14)$
7	$97Y-3X^2=37636$	$(4x_{n+3}-209y_{n+1}, 362y_{2n+2}-6x_{2n+4}+194)$
8	$3Y-X^2=48$	$(y_{n+2}-13y_{n+1}, 15y_{2n+2}-y_{2n+3}+8)$
9	$42Y-X^2=9408$	$(y_{n+3}-181y_{n+1}, 209y_{2n+2}-y_{2n+4}+112)$
10	$4Y-3X^2=16$	$(209x_{n+2}-15x_{n+3}, 13x_{2n+4}-181x_{2n+3}+2)$
11	$Y-3X^2=4$	$(52x_{n+2}-15y_{n+2}, 26y_{2n+3}-90x_{2n+3}+2)$
12	$7Y-3X^2=196$	$(724x_{n+2}-15y_{n+3}, 26y_{2n+4}-1254x_{2n+3}+14)$
13	$7Y-3X^2=196$	$(52x_{n+3}-204y_{n+2}, 362y_{2n+3}-90x_{2n+4}+14)$
14	$Y-3X^2=4$	$(724x_{n+3}-209y_{n+3}, 362y_{2n+4}-1254x_{2n+4}+2)$
15	$3Y-X^2=48$	$(13y_{n+3}-181y_{n+2}, 209y_{2n+3}-15y_{2n+4}+8)$

Some special cases of the solutions are given below.

$$P_y^{10}(t_{3,x+1})^2=108P_x^6(t_{3,y})^2+4(t_{3,y})^2(t_{3,x+1})^2$$

$$9P_y^6(t_{3,x})^2=12P_x^{10}(t_{3,y+1})^2+4(t_{3,x})^2(t_{3,y+1})^2$$

$$P_y^{10}(t_{3,2x-2})^2=12(6P_{x-1}^4)^2(t_{3,y})^2+4(t_{3,y})^2(t_{3,2x-2})^2$$

$$36P_{y-1}^8(t_{3,x})^2=12P_x^{10}(t_{3,2y-2})^2+4(t_{3,x})^2(t_{3,2y-2})^2$$

$$9P_y^6(t_{3,2x-2})^2=12(36P_{x-1}^8)(t_{3,y+1})^2+4(t_{3,2x-2})^2(t_{3,y+1})^2$$

$$(6P_{y-1}^4)^2(t_{3,x+1})^2=12(3P_x^3)^2(t_{3,2y-2})^2+4(t_{3,x+1})^2(t_{3,2y-2})^2$$

V. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the positive Pell Equations $y^2=12x^2+4$ As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties.

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